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Geometry and topology of standart group embeddings

- *Introduction:* Let G be a reductive group over the complex numbers (e.g. G could be $Gl(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $Sp(n,\mathbb{C})$, etc.)

Let X be an (irreducible) algebraic variety over $\mathbb{C}$

Def'n: Let X be a $G\times G$-variety

We say that X is an embedding of G if there exists an open orbit $\mathcal{O}_X \simeq G\times G/_{\Delta G} \approx G$ in X.

Examples: Take $X = M_{n\times n}(\mathbb{C})$ and let $G = GL(n,\mathbb{C})$

$G\times G$ acts on X via $(g,h)[x] \longmapsto (gxh^{-1})$

② $PGl(n,\mathbb{C}) \times PGL(n,\mathbb{C})$ acting on $\mathbb{P}^{n^2-1} = M_{n\times n} - \{0\}/\mathbb{C}^*$

General goal: Study projective group embeddings

Def'n: Let M be a reductive monoid with zero. Let $G \subseteq M$ be the group of units.

$$G\times G\times M \longrightarrow M$$
$$((g,h),x) \longmapsto gxh^{-1}$$

Consider the following projective variety: $\mathbb{P}(M) = M-\{0\}/\mathbb{C}^*$

$\mathbb{P}(M)$ is called an standart group embedding.

A recipe to construct standard embeddings is the following:

Let $\rho: G \longrightarrow GL(V)$ be a representation of G

$\dim V = n$

$$GL(V) \subseteq End(V) \setminus \{0\}$$
$$\downarrow$$
$$\mathbb{P}^{n^2-1}$$

Let $G_0 = \pi(\rho(G))$ Zariski

Take $X = \overline{G_0}^{Zar} \subseteq \mathbb{P}^{n^2-1}$

X is a standard embedding.

Objectives:

Describe ① $H^*_{T\times T}(\mathbb{P}(M))$

② $H^*_{G\times G}(\mathbb{P}(M))$

③ $H^*(\mathbb{P}(M))$

$T \subseteq G$ maximal torus

using combinatorial data: i.e. roots, characters and $\mathcal{R} \subseteq \mathcal{M}$ $\uparrow$ finite monoid

Renner monoid $\mathcal{R} = \overline{N_G}^M / T$

Example:

$M = M(n, \mathbb{C})$

$G = GL(n, \mathbb{C})$

$T = D(n, \mathbb{C})$

$W \simeq S_n$

$W = N_G(T)/T$ — $N_G(T)$: normalizer of T in G

$W \subseteq \mathcal{R} = \{$0-1 matrices with at most one non-zero entry in each row and column$\}$

Objective:

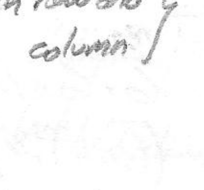
① Describe $H_{T\times T}^*(\mathbb{P}(M))$

② Need to obtain the information about the $T\times T$-fixed points and $T\times T$-invariant curves.

(because $H_{T\times T}^*(\mathbb{P}(M)) \hookrightarrow H_{T\times T}^*(\mathbb{P}(M)^{T\times T})$ and the image can be described in terms of the invariant curves) — $\mathbb{P}(M)^{T\times T}$: fixed point set

"GKM-theory" [Need to find the characters explicitely]

$T \curvearrowright \mathbb{P}^1$ — T acts on $\mathbb{P}^1$ through a character $\chi : T \to \mathbb{C}^*$



② holds for "rationally smooth" standart embeddings.

A variety X is rationally smooth if $\forall x \in X$,

$$H^2(X, X-\{x\}) = \begin{cases} 0 & \text{if } i \neq 2\dim_{\mathbb{C}} X \\ \mathbb{Q} & \text{if } i = 2\dim X \end{cases}$$

$m = |\mathbb{P}(M)^{T\times T}|$

$$H^*_{T\times T}(\mathbb{P}(M)) = \Big\{ (f_1,\dots,f_m) \in \bigoplus_{i=1}^{m} H^*_{T\times T} \;\Big|\; f_i \equiv f_j \bmod \chi_{i,j}$$

whenever x_i and x_j are two fixed points joined by a $T\times T$-curve $C_{i,j}$ on which T acts through $\chi_{i,j}$ $\Big\}$

<u>Theorem:</u> Let $X = \mathbb{P}(M)$ be a rationally smooth standard embedding.

① $H^*_{T\times T}(\mathbb{P}(M)) \hookrightarrow H^*_{T\times T}(\mathbb{P}(M)^{T\times T})$

② Let $\{GeG\}_{e\in\Lambda_1}$ be the set of $G\times G$-closed orbits.

then there is an injection:

$$H^*_{T\times T}(X) \hookrightarrow \bigoplus_{e\in\Lambda_1} H^*_{T\times T}(GeG)$$

in terms of roots

$\forall e\in\Lambda_1$: $GeG \simeq G/P_e \times G/P_{\bar{e}}$

Moreover:

$(\varphi_e) \in \bigoplus_{e\in\Lambda_1} H^*_{T\times T}(GeG)$ satisfies:

① If $f\in E_2(\overline{T})$ and $H_f = \{f, s_{\alpha_f} f\}$

$$\varphi_{e_f}(f_1 u) \equiv \varphi_{e_f}(f_2 u) \bmod (\alpha_f, \alpha_f \circ \mathrm{int}\, u) \quad \text{where } u\in W$$

Γ $f\in E_2(\overline{T})$ $\dim T.f = 2$

$\boxed{H_f = \{f.u \mid u \in C_W(f)\}}$ $\quad f\overline{T} - \{0\}/\mathbb{C}^* \simeq \mathbb{P}^1$

② If $f\in E_2(\overline{T})$ and $H_f = \{f\}$

$$\varphi_{e_f}(f_1 u) \equiv \varphi_{e_f}(f_2 u) \bmod (\lambda_f, \lambda_f \circ \mathrm{int}(u))$$

__Outline:__

* Group actions
* Cohomology, in particular cohomology of S^n, CW-complexes
* Vector bundles (Some characteristic classes)
* Fibre bundles
* Classifying spaces.

__Def'n:__ Let G be a (topological) group. We say that X is a G-space if there is a homomorphism

$$\Psi : G \longrightarrow \underset{\substack{\uparrow \\ \text{(group under composition)}}}{Aut(X)}$$

X variety $\Rightarrow Aut(X) = \{\text{algebraic automorphisms}\}$

X top. space $\Rightarrow Aut(X) = \{\text{homeomorphisms}\}$

$\vdots$

X set $\Rightarrow Aut(X) = \{\text{set of bij. } X \to X\}$

$\boxed{\text{homomorphism } f : G \to H \text{ (groups) satisfy } f(g \cdot_G g') = f(g) \cdot_H f(g')}$

Given $x \in X$ (G-space) we can consider the "orbit" of x denoted $G.x$

$$G.x = \{gx \mid g \in G\}$$

$g \in G$

$g : X \longrightarrow X$

$$\begin{tikzcd} & gx \arrow[r, bend left, "h'"] & h'gx \\ x \arrow[ur, bend left, "g"] \arrow[dr, bend right, "h"] & & \\ & hx & \end{tikzcd}$$

Stabilizer (or isotropy group) of x: is given by

$$Stab_G(x) = \{g \in G \mid g.x = x\}$$

orbit space: $X/G \underset{\substack{\uparrow \\ \text{top. space}}}{=} \{G.x \mid x \in X\}$

Examples:

① $\mathbb{C}^* \times \mathbb{C}^n - \{0\} \longrightarrow \mathbb{C}^n - \{0\}$

$(t,(v_1,\dots,v_n)) \longmapsto (tv_1,\dots,tv_n)$

orbits $\longleftrightarrow$ lines through the origin in $\mathbb{C}^n$

Given $x=(x_1,\dots,x_n)\in\mathbb{C}^n$, $Stab_{\mathbb{C}^*}(x)=\{1\}$

$$\mathbb{C}^n-\{0\}/\mathbb{C}^* \simeq \mathbb{P}^{n-1}$$

② $\mathbb{C}^* \times \mathbb{C}^n - \{0\} \longrightarrow \mathbb{C}^n - \{0\}$

$(t,(v_1,\dots,v_n)) \longmapsto (t^{a_1}v_1,\dots,t^{a_n}v_n)$

$a_i \geq 0$ positive integers.

$\mathbb{P}^{n-1}/\text{finite group}$

$\wr\wr$

Stabilizers are not equal to $\{1\}$ necessarily $\Rightarrow \mathbb{C}^n-\{0\}/\mathbb{C}^* \simeq \mathbb{P}^{n-1}_w$ weighted projective space

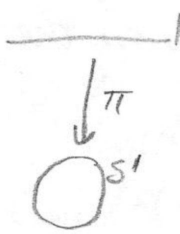
(It has "singularities")

$\mathbb{P}^1_{2,3}$

③ $\mathbb{Z}\times\mathbb{R}\longrightarrow\mathbb{R}$

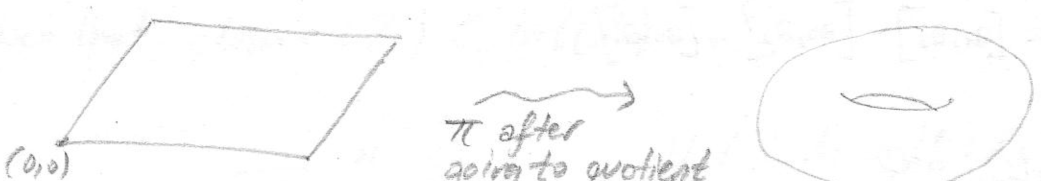
$(n,x)\longmapsto x+n$

$\mathbb{R}/\mathbb{Z}\simeq S^1$



④ $\mathbb{Z}\times\mathbb{Z}\times\mathbb{C}\longrightarrow\mathbb{C}$

$((n,m),(x+iy))\longmapsto(x+n)+i(y+m)$

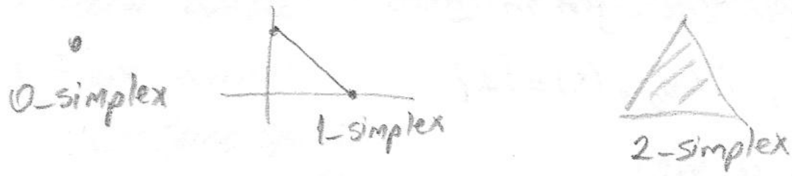


Cohomology

Let X be a topological space.

$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1, \ x_i \geq 0\}$$

n simplex



A map $\sigma : \Delta^n \longrightarrow X$ is called a simplex of X

$\Delta^n \underset{\text{notation}}{=} [a_0, \dots, a_n]$ spanned by n-vertices

notation: $[a_0, \dots, \hat{a}_i, \dots, a_n]$ simplex spanned by all vertices except a_i.

$C_n(X)$ = free abelian group on the n-simplices of X.

Can define a boundary map

$$d : C_n(X) \longrightarrow C_{n-1}(X)$$

$$d\left(\sum_\alpha a_\alpha \sigma_\alpha\right) = \sum_{\alpha, i} (-1)^i a_\alpha \sigma_\alpha [a_0, \dots, \hat{a}_i, \dots a_n]$$

$d\Big($  $\Big) = [a_1, a_2] - [a_0, a_2] + [a_0, a_1]$

$$d_n \circ d_{n+1} = 0$$

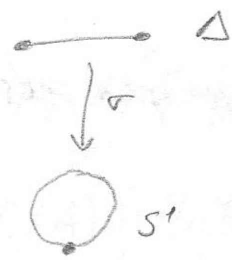
$$C_{n+1}(x) \xrightarrow{d_{n+1}} C_n(x) \xrightarrow{d_n} C_{n-1}(x)$$

Homology groups

$$H^n(X) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$$

Examples

S^1



$\sigma : \Delta^1 \longrightarrow S^1$

It "turns out" $\quad 0 \longrightarrow C_1(X) \xrightarrow{d_1 = 0} C_0(X) \longrightarrow 0$

$H_1(x) = \mathbb{Z}$
$H_0(x) = \mathbb{Z}$

$H_0\Big($  $\Big) = \mathbb{Z} \oplus \mathbb{Z}$

$\text{rank}\big(H_0(X)\big) = \#$ of connected components

Properties:

① $H_0(pt) = \mathbb{Z}$, $H_k(pt) = (0)$

② Excision:

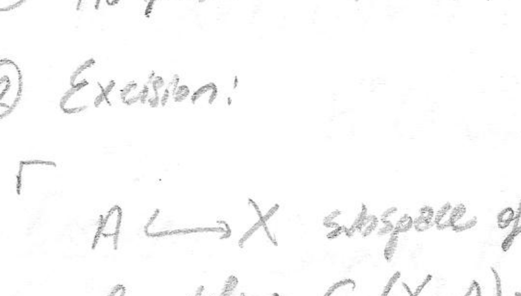
$A \hookrightarrow X$ subspace of X

Can define $C_n(X, A) = C_n(X) / C_n(A)$

$d_n : C_n(X, A) \longrightarrow C_{n-1}(X, A)$

$Z \subseteq A \subseteq X$

Such that closure $(Z) \subseteq \text{int}(A)$



$H_i(X, A) \simeq H_i(X - Z, A - Z)$

equivalently:

$X = A \cup B \quad A, B$ open:

$H_i(X, A) \simeq H_i(B, B \cap A)$

③ Mayer–Vietoris sequence:

$X = A \cup B$ union of two open sets.

there is a long exact sequence:

$$\cdots \longrightarrow H_i(A\cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(X) \longrightarrow H_{i-1}(A\cap B) \longrightarrow H_{i-1}(A) \oplus H_{i-1}(B) \longrightarrow \cdots$$

Can consider

$H_n(X;\mathbb{Q})$ (rational homology)

and cohomology: $H^n(X;\mathbb{Q}) \underset{\text{definition}}{=} \mathrm{Hom}_{\mathbb{Q}}(H_n(X);\mathbb{Q})$

$$C^n(X) := \mathrm{Hom}(C_n(X);\mathbb{Z})$$

$$\cdots \longrightarrow C^n(X) \xrightarrow{\partial_n} C^{n+1}(X) \longrightarrow \cdots$$

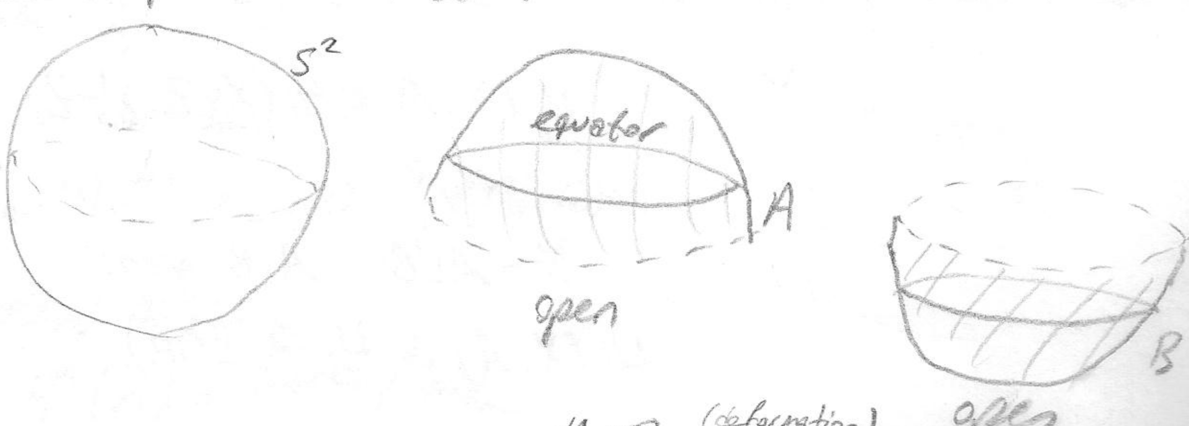
④ Long exact sequence of a pair (X, A):

$$\cdots \longrightarrow H_i(A) \longrightarrow H_i(X) \longrightarrow H_i(X,A) \xrightarrow{\delta} H_{i-1}(A) \longrightarrow \cdots$$

in cohomology

$$\cdots \longrightarrow H^i(X,A) \longrightarrow H^i(X) \longrightarrow H^i(A) \xrightarrow{\delta} H^{i+1}(X,A) \longrightarrow \cdots$$

<u>Example</u>: Compute the homology of S^2



$A \cap B =$ (deformation) retracts to S^1

By Mayer-Vietoris:

$$H_2(A\cap B)^{0} \longrightarrow H_2(A)^{0} \oplus H_2(B)^{0} \longrightarrow H_2(S^2) \to$$

$$\to H_1(\underbrace{A\cap B}_{S^1}) \to H_1(A)^{0}\oplus H_1(B)^{0} \to H_1(S^2) \to H_0(A\cap B) \to \dots$$

$\longrightarrow$ Eksik ?

$$H^k(S^n) = \begin{cases} \mathbb{Z} & k=0,\dots,n \\ 0 & k\neq 0,n \end{cases}$$

Künneth formula

$$H^*(X\times Y;\mathbb{Q}) = H^*(X;\mathbb{Q}) \otimes H^*(Y;\mathbb{Q})$$

$$H^n(X\times Y) = \bigoplus_{i+j=n} H^i(X)\otimes H^j(Y)$$

Exercise: Compute $H^*(S^1\times S^1)$